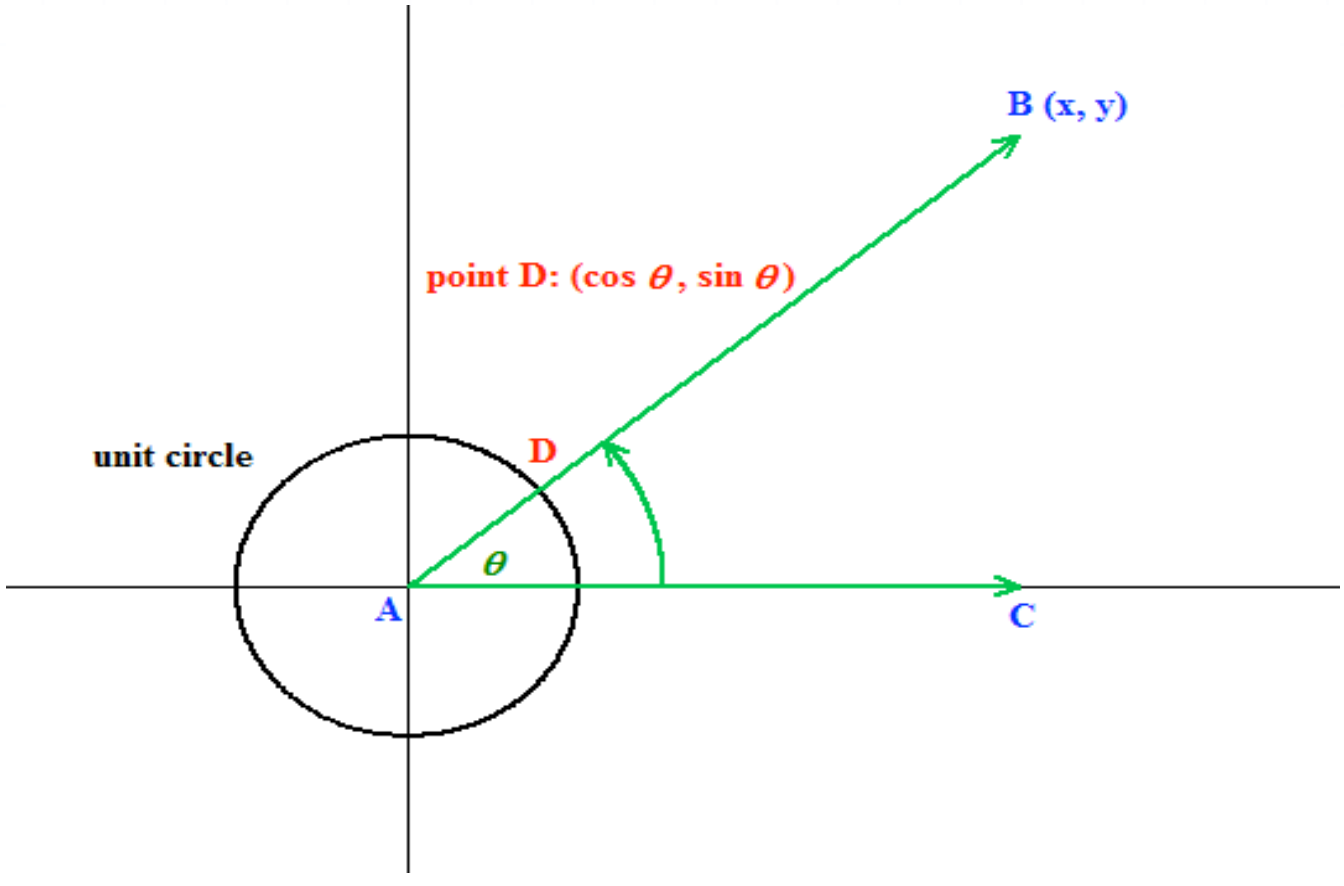
Given a point on the terminal side of the angle

What we want to show in this lesson is that, given any point on the terminal side of an angle, you can find the value of sine and cosine for that angle. And that point doesn't have to be along the unit circle in order for us to find sine and cosine.

To show why this is true, let's say we have a coordinate point B sitting somewhere outside the unit circle and we want to find sine and cosine of the angle θ that's created.

What we want to do is sketch out the unit circle, and label a point D that's on the terminal side of the angle θ created by B, and where the point D is on the unit circle.



Next we'll connect B and D to the x-axis with vertical lines, and label the new points C and E, where C is at $(x,0)$ and E is at $(\cos\theta,0)$.

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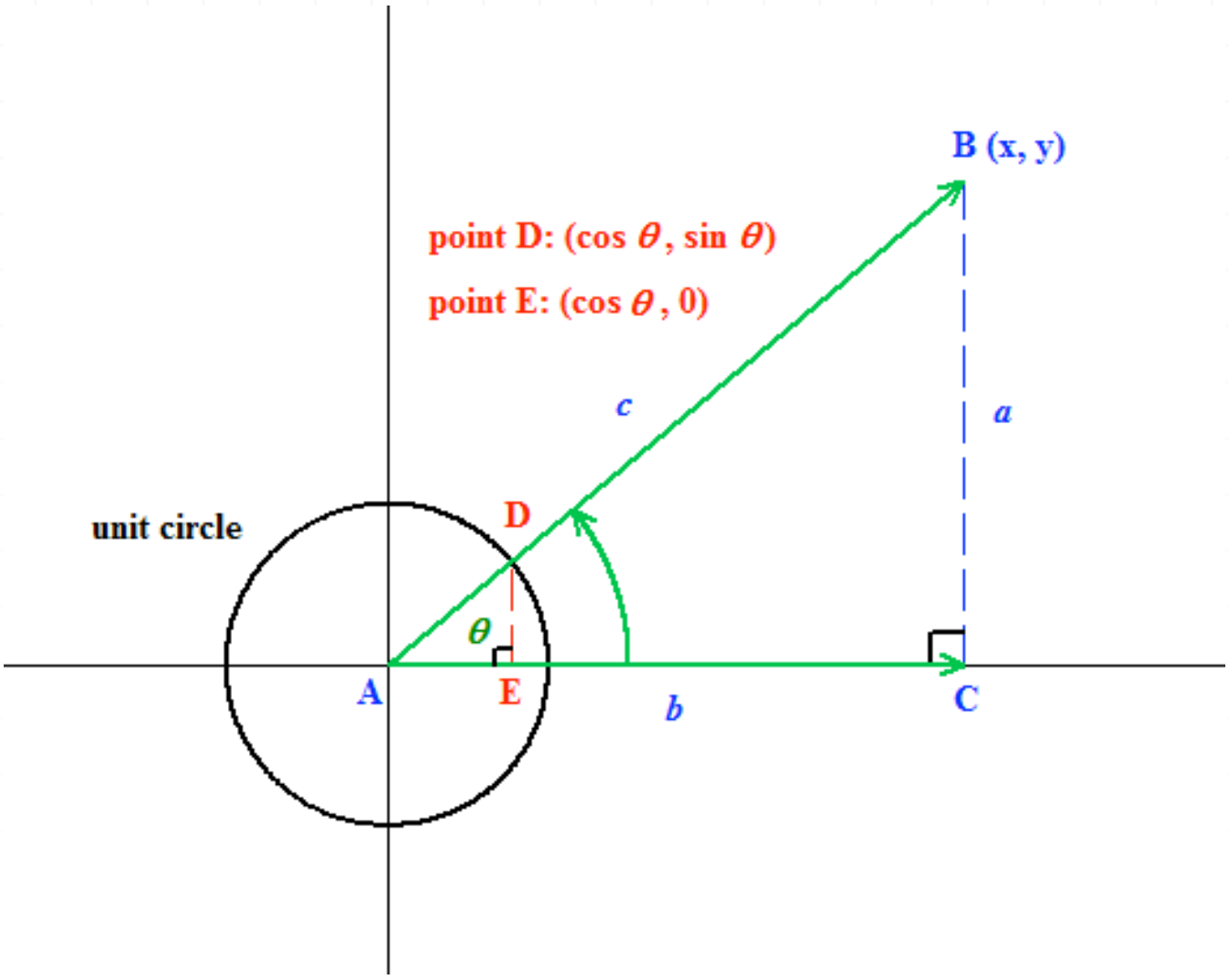
This gives us the two right triangles ABC and ADE, which are similar triangles. Since D is on the unit circle, we know that $\overline{AD} = 1$.



By similarity of right triangles ABC and ADE,

$$\frac{a}{\overline{\text{DE}}} = \frac{b}{\overline{\text{AE}}} = \frac{c}{\overline{\text{AD}}}$$

Since θ is in the first quadrant (which means x and y are both positive), we can say

$$a = y$$

$$b = x$$

By the Pythagorean theorem,

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$$c = \sqrt{a^2 + b^2} = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2}$$

Since θ is in the first quadrant (which means both $\sin\theta$ and $\cos\theta$ are positive), we get

$$\overline{\text{DE}} = \sin\theta$$

$$\overline{\text{AE}} = \cos\theta$$

Substituting these results into our triangle similarity statement, we get

$$\frac{y}{\sin\theta} = \frac{x}{\cos\theta} = \frac{\sqrt{x^2 + y^2}}{1}$$

$$\frac{y}{\sin\theta} = \frac{x}{\cos\theta} = \sqrt{x^2 + y^2}$$

Taking the first and last parts together gives

$$\frac{y}{\sin\theta} = \sqrt{x^2 + y^2}$$

$$\frac{y}{\sqrt{x^2 + y^2}} = \sin\theta$$

Taking the middle and last parts together gives

$$\frac{x}{\cos\theta} = \sqrt{x^2 + y^2}$$

$$\frac{x}{\sqrt{x^2 + y^2}} = \cos\theta$$

Now we've got generic formulas for $\sin\theta$ and $\cos\theta$, which are

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$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

These formulas hold for angles in all four quadrants, so let's look at some examples.

Example

Find the sine and cosine of an angle whose terminal side contains the point $(3,4)$.

The point $(3,4)$ is in the first quadrant, so we know we can use $x = 3$ and $y = 4$ in these formulas:

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

We get

$$\sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

so

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$$\sin\theta = \frac{4}{5}$$

$$\cos\theta = \frac{3}{5}$$

Let's try an example with an angle in the second quadrant.

Example

Find the sine and cosine of an angle whose terminal side contains the point $(-2,3)$.

The point $(-2,3)$ is in the first quadrant, so we know we can use $x = -2$ and $y = 3$ in these formulas:

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

We get

$$\sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

so

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$$\sin\theta = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\cos\theta = -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$$

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